Differential calculus on the quantum superplane

1991 J. Phys. A: Math. Gen. 24 619

(http://iopscience.iop.org/0305-4470/24/3/020)

J. Phys. A: Math. Gen. **24** (1991) 619–624. Printed in the UK

Differential calculus on the quantum superplane

S K Soni

Physics Department, SGTB Khalsa College, University of Delhi, Delhi-110007, India

Received 5 September 1990

Abstract. Very recently Wess and Zumino succeeded in formulating a perfectly consistent differential calculus on the quantum hyperplane. We give the natural extension of their scheme to superspace and discuss the various consistency checks that have been performed.

It is well known, through the work of Woronowicz [1], that quantum groups provide a concrete example of non-commutative differential geometry. In a further development Wess and Zumino [2] have given a simpler example of non-commutative differential geometry. They have shown that one can define a consistent differential calculus on the non-commutative space of the quantum hyperplane. In this paper we consider the extension to superspace of the explicit commutation relations given by Wess and Zumino (see section 4 in [2]). We have checked that the calculus thus developed cannot lead to inconsistencies.

The quantum plane is defined, according to Manin [3], in terms of n coordinates x^i, $i=1,2,\ldots,n$, which satisfy the commutation relations

$$x^i x^j - q x^j x^i = 0 \qquad i<j \tag{1}$$

where q is a complex number (Manin uses $1/q$ where we use q, following the usage of Wess and Zumino). To establish a differential calculus on the quantum (hyper-)plane, Wess and Zumino introduce the differentials of the basic coordinates

$$X^i = \mathrm{d}x^i \tag{2}$$

as additional variables which satisfy $(i<j)$

$$X^i X^j = -(1/q) X^j X^i \tag{3}$$

and also

$$X^i X^i = 0 \qquad \text{for all } i.$$

They deduce the 'intermediary' commutation relations between x's and X's, and also the remaining commutation relations between the now non-commuting derivatives of the quantum plane with the x's, the X's and finally themselves. In deriving these explicit relations they are guided by the requirements of consistency and of covariance under the action of the quantum group $GL_q(n)$. It is known, through the work of Manin and collaborators, that the relations (1) and (3) are preserved under the action of a $\mathrm{GL}_q(n)$ matrix whose elements commute with x's and X's and, vice versa, that

0305-4470/91/030619+06$03.50

this requirement defines a quantum matrix. Wess and Zumino showed that their entire calculus is 'invariant' under the action of this quantum group, in the same sense as (1) and (3) are.

In this work we give explicitly the commutation relations of the differential calculus on the quantum plane with variables x^i, $i=1,2,\ldots,n$, θ^α, $\alpha=1,2,\ldots,m$, which satisfy, in addition to the well known relations (1), the following

$$x^i\theta^\alpha - q\theta^\alpha x^i = 0 \tag{4}$$

and

$$\theta^\alpha\theta^\beta + q\theta^\beta\theta^\alpha = 0 \qquad \alpha<\beta. \tag{5}$$

We also have $\theta^\alpha\theta^\alpha=0$ for each α. Note that for $q=1$, x's commute and θ's anti-commute with themselves while x's commute with θ's. We also introduce variables Θ^α which must satisfy

$$X^i\Theta^\alpha = (1/q)\Theta^\alpha X^i \tag{6}$$

and

$$\Theta^\alpha\Theta^\beta = (1/q)\Theta^\beta\Theta^\alpha. \tag{7}$$

We shall call the plane, defined in terms of the variables satisfying (1), (3)–(7), the quantum superplane. For $q=1$, θ's become Grassmannian in character and commute among themselves. It is natural to ask whether we can interpret Θ's as the differentials of θ's

$$\Theta^\alpha = \mathrm{d}\theta^\alpha \tag{8}$$

in the same manner as X's can be interpreted as differentials of x's.

To develop the differential calculus it is first necessary to give the (anti-)commutation relations between the basic variables and their differentials. We are guided by the following observation: if one differentiates the left-hand sides of the basic relations (1), (4) and (5), then one must require that the result vanish as a consequence of the relations in the calculus. A valid set of relations which respects this 'linear consistency condition' is $(i<j)$

$$x^jX^i = qX^ix^j \tag{9}$$

$$x^iX^j = qX^jx^i + (q^2-1)X^ix^j \tag{10}$$

and $(\alpha<\beta)$

$$\theta^\beta\Theta^\alpha = q\Theta^\alpha\theta^\beta \tag{11}$$

$$\theta^\alpha\Theta^\beta = q\Theta^\beta\theta^\alpha - (q^2-1)\Theta^\alpha\theta^\beta \tag{12}$$

and

$$\theta^\alpha X^i = -qX^i\theta^\alpha \tag{13}$$

$$x^i\Theta^\alpha = q\Theta^\alpha x^i + (q^2-1)X^i\theta^\alpha. \tag{14}$$

The relations (9), (11) and (13) are chosen for compatibility with (3), (6) and (7). (There is a different choice which also implies (3), (6) and (7); for this paper we stick to the above choice.) To the above we add

$$x^i X^i = q^2 X^i x^i \tag{15}$$

and

$$\theta^\alpha \Theta^\alpha = \Theta^\alpha \theta^\alpha. \tag{16}$$

Equation (15) is required for the relations (9), (10) and (15) to be invariant under the action of the quantum group $\mathrm{GL}_q(n)$; as explained by Arne Schirrmacher these relations can be expressed in a simple form (in terms of the R matrix of $\mathrm{GL}_q(n)$) which is covariant under the action of this quantum group. As a further check on our previous relations, let us note that the basic relations (1), (4) and (5) are also valid when acted on by the differentials. It is straightforward, though tedious, to verify that their left-hand sides operating on X's and Θ's lead to a null result. Hence our relations so far are also compatible with the 'quadratic consistency condition' just explained.

The exterior differential (summation over repeated indices)

$$\mathrm{d} = X^i \partial_i + \Theta^\alpha \partial_\alpha \tag{17}$$

has the usual properties such as

$$\mathrm{d}x^i - x^i \mathrm{d} = X^i \tag{18}$$

$$\mathrm{d}\theta^\alpha + \theta^\alpha \mathrm{d} = \Theta^\alpha \tag{19}$$

and $\mathrm{d}^2 = 0$. We have denoted the derivatives

$$\partial_i = \partial/\partial x^i \qquad \partial_\alpha = \partial/\partial\theta^\alpha. \tag{20}$$

From (9)-(19) we arrive at the relations between variables and derivatives

$$\begin{aligned}
&\partial_i x^i = 1 + q^2 x^i \partial_i + (q^2-1) \sum_{i+1}^{n} x^j \partial_j + (q^2-1) \sum_{1}^{m} \theta^\alpha \partial_\alpha \\
&\partial_i x^j = q x^j \partial_i \qquad i<j \\
&\partial_j x^i = q x^i \partial_j \\
&\partial_\alpha x^i = q x^i \partial_\alpha \\
&\partial_i \theta^\alpha = q \theta^\alpha \partial_i \\
&\partial_\alpha \theta^\alpha = 1 - \theta^\alpha \partial_\alpha - (1-q^2) \sum_{\alpha+1}^{m} \theta^\beta \partial_\beta \\
&\partial_\alpha \theta^\beta = -q \theta^\beta \partial_\alpha \qquad \alpha<\beta \\
&\partial_\beta \theta^\alpha = -q \theta^\alpha \partial_\beta.
\end{aligned} \tag{21}$$

From (21) we have the (anti-) commutation relations between the derivatives ($i<j$, $\alpha<\beta$)

$$\begin{aligned}
&\partial_i \partial_j = (1/q) \partial_j \partial_i \\
&\partial_i \partial_\alpha = (1/q) \partial_\alpha \partial_i \\
&\partial_\alpha \partial_\beta = -(1/q) \partial_\beta \partial_\alpha.
\end{aligned} \tag{22}$$

We also have $\partial_\alpha \partial_\alpha = 0$. To complete the scheme we give the relations between derivatives and differentials ($i<j$, $\alpha<\beta$)

$$
\begin{aligned}
&\partial_i X^i - 1/q^2 X^i \partial_i = (1/q^2-1)\sum_1^{i-1} X^k\partial_k\\
&\partial_j X^i - 1/q X^i \partial_j = 0\\
&\partial_i X^j - 1/q X^j \partial_i = 0\\
&\partial_\alpha X^i + 1/q X^i \partial_\alpha = 0\\
&\partial_i \Theta^\alpha - 1/q \Theta^\alpha \partial_i = 0\\
&\partial_\alpha \Theta^\alpha - \Theta^\alpha \partial_\alpha = -(1-1/q^2)\sum_1^n x^k\partial_k + (1-1/q^2)\sum_1^{\alpha-1}\Theta^\gamma\partial_\gamma\\
&\partial_\beta \Theta^\alpha - 1/q \Theta^\alpha \partial_\beta = 0\\
&\partial_\alpha \Theta^\beta - 1/q \Theta^\beta \partial_\alpha = 0.
\end{aligned}
\tag{23}
$$

If we multiply the left-hand side by x^r/Θ^γ from the right, we can commute x^r/Θ^γ through to the left using (21) and the inverse of relations (9)-(16). In this way one finds terms which are linear in X and Θ which must cancel separately. This immediately fixes the coefficients of the exchange terms on the left-hand side of (23). The non-vanishing right-hand-side terms are easy to figure out because the exterior derivative d is expected to satisfy

$$\mathrm{d}X^i + X^i\mathrm{d} = 0 \tag{24}$$

$$\mathrm{d}\Theta^\alpha - \Theta^\alpha\mathrm{d} = 0. \tag{25}$$

Note that the somewhat unexpected relations

$$\mathrm{d}\partial_i = q^2\partial_i\mathrm{d} \tag{26}$$

$$\mathrm{d}\partial_\alpha = -q^2\partial_\alpha\mathrm{d} \tag{27}$$

obtained as a consequence of (22) and (23) are, however, quite consistent with the basic requirement for the exterior derivative, i.e. $\mathrm{d}^2=0$. For instance

$$
\begin{aligned}
\mathrm{d}^2 &= \mathrm{d}(X^i\partial_i + \Theta^\alpha\partial_\alpha) = (-X^i\mathrm{d}\partial_i + \Theta^\alpha\mathrm{d}\partial_\alpha)\\
&= q^2(-X^i\partial_i - \Theta^\alpha\partial_\alpha)\mathrm{d} = -q^2\mathrm{d}^2.
\end{aligned}
\tag{28}
$$

Hence d^2 must vanish. To sum up, we have generalized the scheme of Wess–Zumino by establishing a calculus which satisfies the general constraints for a non-commutative differential calculus (the so-called linear and quadratic conditions) together with various other consistency checks one might think of.

Finally, as a non-trivial illustration, we consider the (2+2)-dimensional case ($n=m=2$). We denote the basic variables x, y, θ and ϕ, the differentials

$$\mathrm{d}x = X \qquad \mathrm{d}y = Y \qquad \mathrm{d}\theta = \Theta \qquad \mathrm{d}\phi = \Phi \tag{29}$$

and the derivatives

$$\partial/\partial x = \partial_x \qquad \partial/\partial y = \partial_y \qquad \partial/\partial\theta = \partial_\theta \qquad \partial/\partial\phi = \partial_\phi. \tag{30}$$

The explicit commutation relations of the differential calculus on the (2+2)-dimensional quantum superplane are obtained readily from the formulae given above.

Let us write the relations between the variables

$$
\begin{aligned}
& xy = qyx \\
& x\theta = q\theta x \quad (\text{idem } y) \\
& x\phi = q\phi x \quad (\text{idem } y) \\
& \theta\phi = -q\phi\theta \\
& \theta^2 = \phi^2 = 0.
\end{aligned}
\tag{31}
$$

To save space we shall not give the remaining relations. Suffice to say that the relations between x, y, X, Y, ∂_x, ∂_y agree with those given by Wess and Zumino (equations (4.6)–(4.12) in [2]) if we let $\theta = \phi = \Theta = \Phi = 0$ in our scheme. Using the basic commutation relations (31) one can order in some standard way an arbitrary monomial (a product of the basic variables elevated to arbitrary powers). For instance one can order monomials so that y appears before x, ϕ is before θ, and x is before ϕ. One can then compute the derivatives of an ordered monomial, using the relations between variables and derivatives. One finds:

$$
\begin{aligned}
& \partial_y(y^n x^m) = (qy)^{n-1} x^m [n] && \partial_x(y^n x^m) = (qy)^n (qx)^{m-1}[m] \\
& \partial_y(y^n x^m \theta) = (qy)^{n-1} x^m \theta [n] && \partial_x(y^n x^m \theta) = (qy)^n (qx)^{m-1}[m] \\
& \partial_y(y^n x^m \phi) = (qy)^{n-1} x^m \phi [n] && \partial_x(y^n x^m \phi) = (qy)^n (qx)^{m-1}[m] \\
& \partial_y(y^n x^m \phi\theta) = (qy)^{n-1} x^m \phi\theta [n] && \partial_x(y^n x^m \phi\theta) = (qy)^n (qx)^{m-1}[m] \\
& \partial_\theta(y^n x^m \theta) = (qy)^{n-1} (qx)^m && \partial_\phi(y^n x^m \phi) = (qy)^n (qx)^m \\
& \partial_\theta(y^n x^m \phi\theta) = (qy)^n (qx)^m (-q\phi) && \partial_\phi(y^n x^m \phi\theta) = (qy)^n (qx)^m \theta
\end{aligned}
\tag{32}
$$

where we have used the standard notation

$$
[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad [m] = \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{33}
$$

By linearity, these formulae yield the corresponding quantum derivatives of ordered formal power series. Alternatively one could take this as the definition of the derivatives and show that the relations in the calculus follow. In this way one is left with no doubt about the internal consistency of the scheme developed above.

In this paper we have primarily considered the *real* quantum superplane:

$$
\begin{aligned}
& \overline{x^i} = x^i \\
& \overline{\theta^\alpha} = \theta^\alpha.
\end{aligned}
\tag{34}
$$

We may choose the differentials X^i and Θ^α to be real and imaginary respectively

$$
\begin{aligned}
& \overline{X^i} = X^i \\
& \overline{\Theta^\alpha} = -\Theta^\alpha
\end{aligned}
\tag{35}
$$

in which case d is imaginary. Since

$$
\begin{aligned}
& \overline{x^i x^j} = \overline{x^j}\,\overline{x^i} = x^j x^i \\
& \overline{x^i \theta^\alpha} = \overline{\theta^\alpha}\,\overline{x^i} = \theta^\alpha x^i \\
& \overline{\theta^\alpha \theta^\beta} = \overline{\theta^\beta}\,\overline{\theta^\alpha} = \theta^\beta \theta^\alpha
\end{aligned}
\tag{36}
$$

we must have that

$$\bar{q} = q^{-1}. \tag{37}$$

One can then check that the entire scheme goes to itself under complex conjugation, provided one also takes

$$\bar{\partial}_i = -q^{2(n-i-m+1)}\partial_i \qquad i = 1, 2, \ldots, n \tag{38}$$

$$\bar{\partial}_\alpha = +q^{-2(m-\alpha)}\partial_\alpha \qquad \alpha = 1, 2, \ldots, m. \tag{39}$$

The complex conjugation thus defined is an involution since its square is the identity. It is evident that

$$\overline{q^{n-i-m+1}\partial_i} = -q^{n+1-m-i}\partial_i \tag{40}$$

are imaginary and

$$\overline{q^{\alpha-m}\partial_\alpha} = +q^{\alpha-m}\partial_\alpha \tag{41}$$

are real. Returning to the (2+2)-dimensional case we note that ∂_x and $q^{-1}\partial_y$ are purely imaginary, while $q^{-1}\partial_\theta$ and ∂_ϕ are real. If we define

$$\begin{aligned} p_x &= -\mathrm{i}(h/2\pi)\partial_x \qquad & p_y &= -\mathrm{i}(h/2\pi)q^{-1}\partial_y \\ p_\theta &= (h/2\pi)q^{-1}\partial_\theta \qquad & p_\phi &= (h/2\pi)\partial_\phi \end{aligned} \tag{42}$$

the quantities x, y, θ, ϕ, p_x, p_y, p_θ and p_ϕ are real and provide a one-parameter deformation of the quantum mechanical phase space for a system with 2 Bose and 2 Fermi degrees of freedom (equivalent to a two-parameter deformation of the classical phase space).

Acknowledgment

This work is supported by Council of Scientific and Industrial Research, New Delhi, India.

References

[1] Woronowicz S L 1986 Group structure on noncommutative spaces *Fields and Geometry* ed A Jadczyk (Singapore: World Scientific)

[2] Wess J and Zumino B 1990 Convariant differential calculus on the quantum hyperplane *Preprint* CERN-TH-5697/90, LAPP-TH-284/90

[3] Manin Yu I 1988 Quantum groups and non-commutative geometry *Preprint* Montreal University CRM-1561; 1989 *Commun. Math. Phys.* **123** 163